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$\mathbb{Z}_n$-graded Lie rings

Pavel Shumyatsky [a], Antonio Tamarozzi [b], Lawrence Wilson [c,*]

[a] *Department of Mathematics, University of Brasilia, 70910-900 Brasilia – DF, Brazil*
[b] *Department of Exact Sciences, Federal University of Mato Grosso do Sul, PO Box 210, Três Lagoas-MS, 79603-011, Brazil*
[c] *Department of Mathematics, University of Florida, PO Box 118105, 358 Little Hall, Gainesville, FL 32611-8105, USA*

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Abstract

Thompson proved that every finite group with a fixed point free automorphism of prime order is nilpotent and G. Higman proved that the nilpotency class is bounded in terms of the prime alone. Kreknin and Kostrikin produced the first explicit bound by reducing to the problem of bounding the nilpotency class of a $\mathbb{Z}_p$-graded Lie ring L with $L_0 = 0$. Meixner later improved this bound. A step in the proof of Kreknin and Kostrikin is to bound the derived length of $\mathbb{Z}_n$-graded Lie rings L with $L_0 = 0$. In this paper we improve these bounds.

1. Introduction

J. Thompson [13] proved that a finite group with a fixed point free automorphism (one fixing only the identity element) of prime order p is necessarily nilpotent. G. Higman [2] proved that, in such a group, the nilpotency class is bounded above by a function depending on the prime p alone. Higman provided examples of such groups with nilpotency class

* Corresponding author.
E-mail addresses: pavel@mat.unb.br (P. Shumyatsky), zzi@ceul.ufms.br (A. Tamarozzi), larryw@math.ufl.edu (L. Wilson).

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$(p^2 - 1)/4$ and proved that for the prime 5 this is the largest possible class. However, in general, Higman's proof does not produce an explicit bound. It was previously known that the bound achieved in Higman's example is the largest possible for the prime 3 (if the prime is 2, then the group must be abelian) and this was later confirmed by B. Scimemi [9][1] (see also [3]) for the prime 7. Much more recently, M. Favaretto, [1], has used a computer to produce a bound of 118 when $p = 11$ and this is the best known bound in this case.

V.A. Kreknin and A.I. Kostrikin [6,7] were the first to provide explicit upper bounds for the nilpotency class in general. As a part of their proof, the problem is reduced to bounding the nilpotency class of a $\mathbb{Z}_p$-graded Lie ring ($\mathbb{Z}$-algebra) L with $L_0 = 0$. Having done this, the proof has two parts: bound the derived length of the Lie ring and bound the nilpotency class in terms of p and the derived length. Kreknin and Kostrikin proved that a $\mathbb{Z}_n$-graded Lie ring with $L_0 = 0$, where now n is any positive natural number, has derived length at most $2^{n-1} - 1$. We will prove the following improvement.

Theorem A. *If L is a $\mathbb{Z}_n$-graded Lie ring with $L_0 = 0$, then the derived length of L is at most $2^{n-4} + \lfloor \log_2(n-1) \rfloor$. If n is a prime p, then the derived length of L is at most $2^{p-5} + \lfloor \log_2(p-3) \rfloor + 2$.*

Our proof will be for $n \geqslant 8$. The earlier cases were handled by L. Kovács [5], for $n = 4$, by Higman for $n = 5$, by the first two authors [11], for $n = 6$, and by Scimemi [9], for $n = 7$.

Kreknin and Kostrikin bounded the nilpotency class of a $\mathbb{Z}_p$-graded Lie ring L with $L_0 = 0$ using the prime p and the derived length of L. They proved that if the derived length of L is s, then the nilpotency class of L is at most $((p-1)^s - 1)/(p-2)$. Meixner [8] was also able to improve this to $(p-1)^{s-1}$. We prove the following improvement:

Theorem B. *If L is a $\mathbb{Z}_p$-graded Lie ring with $L_0 = 0$ and $L^{(s)} = 0$ then, the nilpotency class of L is at most*

$$\frac{(p-2)^s - 1}{p-3}.$$

The following is a corollary of Theorems A and B.

Theorem C. *If L is a $\mathbb{Z}_p$-graded Lie ring with $L_0 = 0$, then the nilpotency class of L is at most $((p-2)^s - 1)/(p-3)$ where $s = 2^{p-5} + \lfloor \log_2(p-3) \rfloor + 2$.*

It is important to note that these improved bounds are still far larger than those achieved in Higman's example, which remain the largest known to date. Also, it is widely believed that the correct bound on the derived length of a $\mathbb{Z}_n$-graded Lie ring with $L_0 = 0$ is at most n. A general introduction to these questions can be found in [4] and [10]. Some results of this paper appeared in the unpublished works [10] of the first author and [12] of the second author.

[1] These unpublished results are confirmed in [1].

In Section 2 we present the notation and give preliminary results. Section 3 contains the proof of Theorem A and Section 4 that of Theorem B.

2. Notation and preliminaries

Throughout this paper, n is a positive integer at least 8 and p is a prime at least 11. L will be a $\mathbb{Z}_n$-graded or $\mathbb{Z}_p$-graded Lie ring with $L_0 = 0$. The context should always make clear which is intended. We use $[\ell_1, \ell_2, \ell_3] = [[\ell_1, \ell_2], \ell_3]$. If $L_{i_1}, \dots, L_{i_m}$ are not necessarily distinct homogeneous components of L, then we denote by $[L_{i_1}, \dots, L_{i_m}]$ the subgroup of the additive group of L generated by elements of the form $[\ell_1, \dots, \ell_m]$ where each ℓ_j is in L_{i_j}. We use $L^{(k)}$ for the kth term of the derived series of L. If M is a subset of L, then $\langle M \rangle$ denotes for the subring generated by M.

We introduce the terms $L_a^{(0)} = L_a$ and

$$L_a^{(k+1)} = \sum_{b+c=a} \bigl[L_b^{(k)}, L_c^{(k)}\bigr].$$

There is no ambiguity in this notation since the homogeneous components L_a are merely subgroups of the additive group of L and hence the derived series of L_a is not defined. The subring $L^{(k)}$ is $\mathbb{Z}_n$-graded with homogeneous components $L_a^{(k)}$. Notice that $L_a^{(k)} = L_k \cap L(a)$. If, however, L is not a direct sum of the homogenous components, then $L_k \cap L^a \subseteq L_a^{(k)}$, but equality does not always hold. Nevertheless, we do not need to assume that L is a direct sum and all our arguments work in this case too.

The following lemma will be used without reference. Each part is a simple consequence of the hypothesis, the last following from the Jacobi identity in L.

Lemma 2.1. *Let L be a $\mathbb{Z}_n$-graded Lie ring with $L_0 = 0$. Then*:

(a) *If $i_1, \dots, i_m$ are in $\mathbb{Z}_n$ and $i_1 + \cdots + i_m = 0$, then $[L_{i_1}, \dots, L_{i_m}] = 0$.*
(b) *$[L_i, L_j] = [L_j, L_i]$ and $[L_i, L_j, L_{-j}] = [L_i, L_{-j}, L_j]$ for all i and j in $\mathbb{Z}_n$.*
(c) *$[L_i, L_j, L_k] \subseteq [L_{i+k}, L_j] + [L_i, L_{j+k}]$ for all i, j, and k in $\mathbb{Z}_n$.*

We will need one other preliminary lemma.

Lemma 2.2. *Let L be a $\mathbb{Z}_n$-graded Lie ring. Then $[L_a, L_i^{(k)}] \subseteq L_{i+a}^{(k)}$.*

Proof. We proceed by induction on k, the case $k = 0$ following from the fact that L is $\mathbb{Z}_n$-graded. Assume the result for k and let us prove it for $k + 1$. Here

$$\begin{aligned} \Bigl[L_a, \sum_{b+c=i} \bigl[L_b^{(k)}, L_c^{(k)}\bigr]\Bigr] &\subseteq \sum_{b+c=i} \bigl[L_b^{(k)}, L_c^{(k)}, L_a\bigr] \\ &\subseteq \sum_{b+c=i} \bigl[\bigl[L_b^{(k)}, L_a\bigr], L_c^{(k)}\bigr] + \bigl[L_b^{(k)}, \bigl[L_c^{(k)}, L_a\bigr]\bigr] \end{aligned}$$

$$\subseteq \sum_{b+c=i} \left[L_{b+a}^{(k)}, L_c^{(k)}\right] + \left[L_b^{(k)}, L_{c+a}^{(k)}\right] \subseteq L_{i+a}^{(k+1)}.$$

This ends the proof of the lemma. □

For Section 4, we need the following notion: if $a_1, \ldots, a_k$ are elements of $\mathbb{Z}_p$, then we say that they *produce* the elements $\sum_{i \in S} a_i$ where S is any subset of $\{1, \ldots, k\}$. If $S = \emptyset$, we take this sum to be 0. Kreknin and Kostrikin proved the following lemma.

Lemma 2.3. *Let $a_1, \ldots, a_k$ be not necessarily distinct non-zero elements of $\mathbb{Z}_p$. Either they produce at least $k+1$ pairwise distinct elements of $\mathbb{Z}_p$ or they produce all elements of $\mathbb{Z}_p$.*

We will require the following refinement of Lemma 2.3.

Lemma 2.4. *Let $a_1, \ldots, a_k$ be not necessarily distinct non-zero elements of $\mathbb{Z}_p$. If they produce exactly $k+1 < p$ distinct elements of $\mathbb{Z}_p$, then each a_j is equal to either a_1 or $-a_1$.*

Proof. We proceed by induction on k. For $k = 2$, if $a_2 \neq \pm a_1$, then 0, a_1, a_2, and $a_1 + a_2$ are four pairwise distinct elements produced by a_1 and a_2. Assume now that the result holds for $k-1$ and let us prove it for k. Call X_s the set of elements produced by $a_1, \ldots, a_s$. Then $X_k = X_{k-1} \cup (X_{k-1} + a_k)$.

By Lemma 2.3, we know X_{k-1} contains at least k elements or else all of $\mathbb{Z}_p$. Under our hypothesis, therefore, $|X_{k-1}|$ is either k or $k+1$. If the size is $k+1$, then $X_k = X_{k-1}$ and hence X_{k-1} contains the subgroup generated by a_k, which is all of $\mathbb{Z}_p$. We conclude that $|X_{k-1}| = k$ and hence each a_j is $\pm a_1$ for $2 \leqslant j \leqslant k-1$. Repeating the argument using the elements produced by $X_{k-2} \cup \{a_k\}$ yields that also $a_k = \pm a_1$ and this completes the proof. □

3. The derived length of $\mathbb{Z}_n$-graded Lie rings

Our first goal is to prove

Proposition 3.1. *If $n \geqslant 8$ and L is a $\mathbb{Z}_n$-graded Lie ring with $L_0 = 0$, then $L^{(3)} \subseteq \langle L_1, L_5, L_6, \ldots, L_{n-1} \rangle$.*

Let us refer to this subring as M. In order to prove this we need only prove that $L_2^{(3)}$, $L_3^{(3)}$, and $L_4^{(3)}$ are contained in M. This we do in the following three lemmas.

Lemma 3.2. $L_2^{(3)} \subseteq \langle L_1, L_5, L_6, \ldots, L_{n-1} \rangle$.

Proof. If $i + j = 2$ and i and j are in $\{1, 5, 6, \ldots, n-3\}$, then clearly $[L_i^{(2)}, L_j^{(2)}] \subseteq M$. We need only prove then that $[L_3^{(2)}, L_{n-1}^{(2)}]$ and $[L_4^{(2)}, L_{n-2}^{(2)}]$ are contained in M.

We first consider the first of these commutators. Again, if $i + j = 3$ and i and j are in $\{5, 6, \dots, n-2\}$, then $[L_i^{(1)}, L_j^{(1)}, L_{n-1}^{(2)}] \subseteq M$. Hence we need only prove that $[L_2^{(1)}, L_1^{(1)}, L_{n-1}^{(2)}]$ and $[L_4^{(1)}, L_{n-1}^{(1)}, L_{n-1}^{(2)}]$ are contained in M. This first is equal to $[L_2^{(1)}, L_{n-1}^{(2)}, L_1^{(1)}]$ and this is contained in $[L_1, L_1] \subseteq M$. For the latter, we need only consider $[L_2, L_2, L_{n-1}^{(1)}, L_{n-1}^{(2)}]$ and $[L_3, L_1, L_{n-1}^{(1)}, L_{n-1}^{(2)}]$. The second of these is contained in $[L_3, L_{n-1}, L_{n-1}, L_1] \subseteq [L_1, L_1] \subseteq M$. For the first, we note that $[L_2, L_2, L_{n-1}, L_{n-1}] \subseteq [L_1, L_2, L_{n-1}] + [L_2, L_1, L_{n-1}]$. The two summands are equal and $[L_2, L_1, L_{n-1}] = [L_2, L_{n-1}, L_1]$ and this last is contained in $[L_1, L_1] \subseteq M$. We have now proven that $[L_3^{(2)}, L_{n-1}^{(2)}]$ is contained in M.

Similarly, to prove that $[L_4^{(2)}, L_{n-2}^{(2)}] \subseteq M$, we need only confirm that both $[L_3^{(1)}, L_1^{(1)}, L_{n-2}^{(2)}]$ and $[L_2^{(1)}, L_2^{(1)}, L_{n-2}^{(2)}]$ are contained in M. The latter of these is clearly equal to 0 and hence we need only consider the former. In expanding $L_3^{(1)}$, the only terms we need consider are $[L_2, L_1]$ and $[L_4, L_{n-1}]$. We note that

$$[L_2, L_1, L_1, L_{n-2}] \subseteq [L_2, L_1, L_{n-2}, L_1] + [L_2, L_1, L_{n-1}].$$

The first of these is contained in $[L_1, L_1] \subseteq M$ and the second is equal to $[L_2, L_{n-1}, L_1]$ which is also contained in M. Finally, $[L_4, L_{n-1}, L_1, L_{n-2}] \subseteq [L_5, L_{n-1}, L_{n-2}]$ and this is contained in M. This completes the proof of the lemma. □

Lemma 3.3. $L_4^{(3)} \subseteq \langle L_1, L_5, L_6, \dots, L_{n-1}\rangle$.

Proof. As above we need only check that $[L_3^{(2)}, L_1^{(2)}]$ and $[L_2^{(2)}, L_2^{(2)}]$ are contained in M. For the first, we need only check this for $[L_2^{(1)}, L_1^{(1)}, L_1^{(2)}]$ and $[L_4^{(1)}, L_{n-1}^{(1)}, L_1^{(2)}]$. The latter of these is contained in $[L_5, L_{n-1}]$ which is contained in M. For the former, we must confirm that $[L_3, L_{n-1}, L_1^{(1)}, L_1^{(2)}]$ and $[L_4, L_{n-2}, L_1^{(1)}, L_1^{(2)}]$ are contained in M. The first is contained in $[L_5, L_{n-1}]$. The second is contained in $[L_5, L_{n-2}, L_1] + [L_4, L_{n-1}, L_1]$ by the Jacobi identity. The latter of these is clearly contained in $[L_5, L_{n-1}]$ and the former, by the Jacobi identity, is contained in $[L_6, L_{n-2}] + [L_5, L_{n-1}]$. These are all contained in M as $n \geqslant 8$ and so $n - 2 \geqslant 6$.

It only remains for us to prove that $[L_2^{(2)}, L_2^{(2)}]$ is contained in M. We claim that if i and j are in $\{5, 6, \dots, n-1\}$, then $[L_i, L_j, L_2] \subseteq \langle L_5, L_6, \dots, L_{n-1}\rangle$. By the Jacobi identity, this is contained in $[L_{i+2}, L_j] + [L_i, L_{j+2}]$. We are done unless $i + 2$ or $j + 2$ is 1. Taking $i + 2 = 1$ then the whole commutator is contained in L_{j+1} and the result is clear for L_{j+1} unless $j = n - 1$ but then $L_{j+1} = 0$. Therefore, it is enough for us to look at the terms $[L_1^{(1)}, L_1^{(1)}]$, $[L_3^{(1)}, L_{n-1}^{(1)}]$ and $[L_4^{(1)}, L_{n-2}^{(1)}]$. We note that $[L_4^{(1)}, L_{n-2}^{(1)}, L_2^{(2)}] \subseteq [L_6, L_{n-2}]$ and hence is contained in M. Also $[[L_1^{(1)}, L_1^{(1)}], [L_1^{(1)}, L_1^{(1)}]]$ is contained in M. Finally, $[[L_3^{(1)}, L_{n-1}^{(1)}], [L_1^{(1)}, L_1^{(1)}]]$ is contained in $[L_5, L_{n-1}] \subseteq M$.

Therefore, it only remains for us to prove that $[[L_3^{(1)}, L_{n-1}^{(1)}], [L_3^{(1)}, L_{n-1}^{(1)}]]$ is contained in M. In expanding the two $L_3^{(1)}$ we get something that is clearly contained in M unless one of the terms is $[L_2, L_1]$ or $[L_4, L_{n-1}]$. However, $[L_2, L_1, L_{n-1}^{(1)}] \subseteq [L_1, L_1]$. Finally, note that $[L_4, L_{n-1}, L_{n-1}^{(1)}, L_2^{(2)}]$ is contained in $[L_4, L_{n-1}, L_1] + [L_4, L_{n-1}, L_2, L_{n-1}]$ by the Jacobi identity. Both of these are contained in $[L_5, L_{n-1}]$ and this completes the proof. □

Lemma 3.4. $L_3^{(3)} \subseteq \langle L_1, L_5, L_6, \ldots, L_{n-1}\rangle$.

Proof. To prove this we need to prove that $[L_2^{(2)}, L_1^{(2)}]$ and $[L_4^{(2)}, L_{n-1}^{(2)}]$ are in M. As L_1 is in M, for the first of these we need only prove that $[L_3^{(1)}, L_{n-1}^{(1)}, L_1^{(2)}]$ and $[L_4^{(1)}, L_{n-2}^{(1)}, L_1^{(2)}]$ are in M. The first of these is in M as $[L_2, L_1, L_{n-1}, L_1] \subseteq [L_1, L_1, L_1]$ and $[L_4, L_{n-1}, L_{n-1}, L_1] \subseteq [L_5, L_{n-1}, L_{n-1}]$. For the latter, we note that $[L_2, L_2, L_{n-2}, L_1] = 0$ and hence we need only prove that $[L_1, L_3, L_{n-2}^{(1)}, L_1^{(2)}]$ is in M. By the Jacobi identity this is contained in $[L_1, L_1, L_1] + [L_1, L_{n-2}^{(1)}, L_3, L_1^{(2)}]$. The first is clearly contained in M and the second is contained, by the Jacobi identity, in $[L_1, L_{n-2}, L_1, L_3] + [L_1, L_{n-2}^{(1)}, [L_3, L_1^{(2)}]]$. The first of these is 0.

Lemma 2.2 implies that $[L_3, L_1^{(2)}] \subseteq L_4^{(2)}$ and that $[L_1, L_{n-2}^{(1)}] \subseteq L_{n-1}^{(1)}$. Hence it suffices to prove that $[L_4^{(2)}, L_{n-1}^{(1)}] \subseteq M$ to prove that $[L_2^{(2)}, L_1^{(2)}]$ is contained in M. Thus proving this will prove that $L_3^{(3)}$ is contained in M. We claim that

$$L_4^{(2)} \subseteq [L_2, L_1, L_1] + \bigl[[L_3, L_{n-1}], [L_3, L_{n-1}]\bigr] + M.$$

To prove this it is enough to check it for $[L_3^{(1)}, L_1^{(1)}]$ and for $[L_2^{(1)}, L_2^{(1)}]$. For the former, we need only check this for $[L_2, L_1, L_1^{(1)}]$ and for $[L_4, L_{n-1}, L_1]$. The former is included and the latter is contained in $[L_5, L_{n-1}] \subseteq M$. If $i + j = 2$ and $5 \leqslant i, j \leqslant n - 3$, then $[L_i, L_j, L_2] \subseteq [L_{i+2}, L_j] + [L_i, L_{j+2}]$ and both of these are in M. Also, $[[L_1, L_1], L_2^{(1)}] \subseteq [L_2, L_1, L_1]$ and $[[L_4, L_{n-2}], L_2^{(1)}] \subseteq [L_6, L_{n-2}] \subseteq M$. The only term left over in $[L_2^{(1)}, L_2^{(1)}]$ is $[[L_3, L_{n-1}], [L_3, L_{n-1}]]$. This completes the proof of the claim.

As $[L_2, L_1, L_1, L_{n-1}] = [L_2, L_{n-1}, L_1, L_1] \subseteq M$, we need only prove that $[[L_3, L_{n-1}], [L_3, L_{n-1}], L_{n-1}^{(1)}]$ is contained in M. If $i + j = n - 1$ and $5 \leqslant i, j \leqslant n - 6$, then $[L_i, L_j, L_4] \subseteq [L_{i+4}, L_j] + [L_i, L_{j+4}]$ and these are both contained in M. Certainly $[L_4, L_1, L_{n-2}] \subseteq [L_5, L_{n-2}] \subseteq M$ while $[L_2, L_2, L_{n-2}, L_1] = 0$. Thus it remains for us to prove that $[[L_3, L_{n-1}], [L_3, L_{n-1}], [L_a, L_b]]$ is contained in M for $\{a, b\}$ one of $\{2, n-3\}$, $\{3, n-4\}$, and $\{4, n-5\}$.

We can see that $[L_4, L_2, L_{n-3}] \subseteq M$ as $n \geqslant 8$. Note that

$$[L_3, L_{n-1}, L_3, L_{n-1}, L_{n-1}] \subseteq [L_5, L_{n-1}, L_{n-1}] \subseteq M.$$

Therefore, we can consider $[L_3, L_{n-1}, L_{n-1}, L_3, L_{n-3}, L_2]$. This is equal to $[L_3, L_{n-1}, L_{n-1}, L_{n-3}, L_3, L_2]$ and hence contained in $[L_{n-2}, L_3, L_2]$. This last is contained in $[L_5, L_{n-2}] \subseteq M$.

Certainly $[L_4, L_{n-4}, L_3] = 0$ and hence it only remains for us to prove that $[L_3, L_{n-1}, [L_{n-1}, L_3], L_3, L_{n-4}] \subseteq M$. This is contained in $[L_7, L_{n-4}]$ and this is certainly contained in M when $n \geqslant 9$. Taking $n = 8$ we see that $[L_3, L_7, [L_3, L_7], L_3, L_4]$ is contained in $[L_3, L_7, L_3, [L_3, L_7], L_4] + [L_3, L_7, L_5, L_4]$. The latter is equal to $[L_7, L_5, L_3, L_4]$

and this is contained in $[L_5, L_7, L_7] + [L_5, L_7, L_4, L_3]$ and the first is in M and the last is contained in $[L_0, L_3]$. Now $[L_5, [L_3, L_7], L_4]$ is contained in $[L_5, L_3, L_7, L_4] + [L_5, L_7, L_3, L_4]$. Note that $[L_5, L_3] = 0$ and we have already seen that the latter is contained in M.

In the case $n = 8$ the set $\{4, n - 5\}$ is the same as the set $\{3, n - 4\}$ and so we can now assume that $n \geqslant 9$. Certainly $[L_4, L_4, L_{n-5}]$ is contained in M for $n \geqslant 10$. Note that $[L_3, L_{n-1}, L_{n-1}, L_3, L_{n-5}, L_4]$ is contained in $[L_{n-4}, L_3, L_4] + [L_3, L_{n-1}, L_{n-1}, L_{n-2}, L_4]$. The former is in M. The latter is equal to $[L_{n-1}, L_{n-2}, [L_3, L_{n-1}], L_4]$. This is contained in $[L_{n-3}, L_3, L_{n-1}, L_4] + [L_{n-3}, L_{n-1}, L_3, L_4]$. The former of these is 0 and the latter is contained in M. For $n = 9$ we have $4 = n - 5$ and so we have finished that case as well. $\square$

We now wish to use Proposition 3.1 to find information about higher terms of the derived series of L. We begin with a technical lemma.

Lemma 3.5. *If $k + 1 \leqslant i \leqslant n - 1$ and $[L_j, L_i, \overbrace{L_1, \ldots, L_1}^{r}] \subseteq L_k$, then*

$$[L_j, L_i, \overbrace{L_1, \ldots, L_1}^{r}] \subseteq \langle L_{k+1}, \ldots, L_{n-1} \rangle.$$

Proof. We proceed by induction on r. If $r = 0$, then $i + j \equiv k$ modulo n and, as $i \geqslant k + 1$, we conclude that $j \geqslant k+1$ and hence $[L_i, L_j] \subseteq \langle L_{k+1}, \ldots, L_{n-1} \rangle$. Now assume the result for r and let us prove it for $r + 1$.

$$[L_j, L_i, L_1, \overbrace{L_1, \ldots, L_1}^{r}] \subseteq [L_{j+1}, L_i, \overbrace{L_1, \ldots, L_1}^{r}] + [L_j, L_{i+1}, \overbrace{L_1, \ldots, L_1}^{r}].$$

By the induction hypothesis, these are both in $\langle L_{k+1}, \ldots, L_{n-1} \rangle$ unless $i + 1 = n$ but then that term is 0. $\square$

This next lemma mimics an argument of Kreknin and Kostrikin.

Lemma 3.6. *If $L^{(m)} \subseteq \langle L_1, L_k, L_{k+1}, \ldots, L_{n-1} \rangle$, then*

$$L_k^{(m+1)} \subseteq \langle L_1, L_{k+1}, L_{k+2}, \ldots, L_{n-1} \rangle.$$

Proof. In expanding $L_k^{(m+1)}$ each term is of the form $L_i^{(m)}$ and hence we can express it as commutators involving the terms L_1, L_k, $L_{k+1}, \ldots, L_{n-2}$, and L_{n-1}. Using the Jacobi identity to get left-normed commutators from $[L_i^{(m)}, L_j^{(m)}]$ we get commutators of the form $[L_{a_1}, \ldots, L_{a_s}]$ with each a_j in $\{1, k, k + 1, \ldots, n - 1\}$.

If each $a_j = 1$, then we are done. Otherwise, let t be the largest index such that $a_t \neq 1$. If $a_t \geqslant k + 1$, then Lemma 3.5 implies that this commutator lies in $\langle L_{k+1}, L_{k+2}, \ldots, L_{n-1} \rangle$. That leaves only the case that $a_t = k$. We proceed by reverse induction on t.

If $t = s$, then $[L_{a_1}, \dots, L_{a_{s-1}}] \subseteq L_0 = 0$ and hence the result holds in this case. Assume now the result for $t + 1$ and let us prove it for $t < s$. We have that $[L_{a_1}, \dots, L_{a_s}]$ is

$$[L_{a_1}, \dots, L_{a_{t-1}}, L_k, L_1, \overbrace{L_1, \dots, L_1}^{s-t-1}]$$
$$\subseteq [L_{a_1}, \dots, L_{a_{t-1}}, L_1, L_k, \overbrace{L_1, \dots, L_1}^{s-t-1}] + [L_{a_1}, \dots, L_{a_{t-1}}, L_{k+1}, \overbrace{L_1, \dots, L_1}^{s-t-1}].$$

That the former lies in the desired ideal is the induction hypothesis and that the latter does is Lemma 3.5. This ends the proof. □

We are now prepared to get information about higher terms of the derived series of L.

Lemma 3.7. *If every $\mathbb{Z}_n$-graded Lie ring L with $L_0 = 0$ satisfies*

$$L^{(m)} \subseteq \langle L_1, L_k, L_{k+1}, \dots, L_{n-1}\rangle,$$

then they also satisfy

$$L^{(2m+1)} \subseteq \langle L_1, L_{k+1}, L_{k+2}, \dots, L_{n-1}\rangle.$$

Proof. Set $H = L^{(m+1)}$. Then H is also $\mathbb{Z}_n$-graded as discussed and so $L^{(2m+1)} = H^{(m)} \subseteq \langle H_1, H_k, H_{k+1}, \dots, H_{n-1}\rangle$. Lemma 3.6 implies that $H_k \subseteq \langle L_1, L_{k+1}, L_{k+2}, \dots, L_{n-1}\rangle$ and each $H_i \subseteq L_i$ and so we conclude that $L^{(2m+1)} \subseteq \langle L_1, L_{k+1}, L_{k+2}, \dots, L_{n-1}\rangle$. □

Multiple applications of Lemma 3.7 to Proposition 3.1 imply the following.

Corollary 3.8. *If L is a $\mathbb{Z}_n$-graded with $L_0 = 0$, then $L^{(2^{n-4}-1)} \subseteq \langle L_1, L_{n-1}\rangle$.*

We are now ready to prove Theorem A.

Proof of Theorem A. From Corollary 3.8 we know that $L^{(2^{n-4}-1)}$ is contained in $\langle L_1, L_{n-1}\rangle$. As L_1 and L_{n-1} commute, this last is the same as $\langle L_1\rangle \oplus \langle L_{n-1}\rangle$. Clearly each $\langle L_k\rangle$ has nilpotency class at most $n - 1$ and hence this direct sum has derived length at most $\lfloor \log_2(n-1)\rfloor + 1$. Therefore, $L^{(2^{n-4}+\lfloor \log_2(n-1)\rfloor)} = 0$, as desired.

Now assume that n is a prime p. As in the proof of Corollary 3.8, we deduce that $L^{(2^{p-5}-1)}$ is contained in $\langle L_1, L_{p-2}, L_{p-1}\rangle$. It is clear that the quotient $\langle L_1, L_{p-2}, L_{p-1}\rangle/\langle L_{p-2}, L_{p-1}\rangle$ has nilpotency class at most $p-3$ and hence $L^{(2^{p-5}+\lfloor \log_2(p-3)\rfloor)}$ is contained in $\langle L_{p-2}, L_{p-1}\rangle$. In $L_{p-2}^{(2^{p-5}+\lfloor \log_2(p-3)\rfloor+1)}$, we expand each term out in terms of L_{p-1} and L_{p-2} and then express this as a sum of left-normed commutators. If the last term is L_{p-2}, then the previous terms give a commutator in L_0. If the last term is L_{p-1}, then the previous terms give a commutator in L_{p-1} and hence we deduce that $L_{p-2}^{(2^{p-5}+\lfloor \log_2(p-3)\rfloor+1)}$ is contained in $\langle L_{p-1}\rangle$. Hence $L_{p-2}^{(2^{p-5}+\lfloor \log_2(p-3)\rfloor+1)}$ commutes with both L_1 and L_2.

Using the grading $L_{p-1} + L_{p-2} + \cdots + L_1$, we find $L^{(2^{p-5}+\lfloor \log_2(p-3)\rfloor)}$ is contained in $\langle L_2, L_1\rangle$ and hence $L_{p-2}^{(2^{p-5}+\lfloor \log_2(p-3)\rfloor+1)}$ is contained in the center of $L^{(2^{p-5}+\lfloor \log_2(p-3)\rfloor)}$. In the case of a prime, if k is any term between 1 and $p-1$, then we can give a new grading for L via $L_{k/(p-2)} + L_{2k/(p-2)} + \cdots + L_{(p-1)k/(p-2)}$ in which case the $(p-2)$nd term is L_k. Therefore each $L_k^{(2^{p-5}+\lfloor \log_2(p-3)\rfloor+1)}$ commutes with $L^{(2^{p-5}+\lfloor \log_2(p-3)\rfloor)}$. Thus $L^{(2^{p-5}+\lfloor \log_2(p-3)\rfloor+1)}$ is contained in the center of $L^{(2^{p-5}+\lfloor \log_2(p-3)\rfloor)}$ and this yields the result. $\square$

4. The nilpotency class of $\mathbb{Z}_p$-graded Lie rings

We are now only going to consider the case of a prime p. We wish to prove Theorem B. We first produce a technical lemma showing that certain commutators in the L_a are trivial.

Lemma 4.1. *If L is a $\mathbb{Z}_p$-graded Lie ring with $L_0 = 0$ and a, b, and $c_1, \ldots, c_{p-2}$ are non-zero terms in $\mathbb{Z}_p$ where each $c_i = \pm c_1$, then $[L_a, L_b, L_{c_1}, \ldots, L_{c_{p-2}}] = 0$.*

Proof. As L_{c_1} and L_{-c_1} commute, this is certainly true if $c_1, \ldots, c_{p-2}$ produce $-(a+b)$. If $b = \pm c_1$, then the $p-1$ such terms can generate all of $\mathbb{Z}_p$, by Lemma 2.3, including $-a$ and hence the commutator is 0. As $[L_a, L_b] = [L_b, L_a]$ we have reduced to the case that neither a nor b is $\pm c_1$.

If there are k of the c_i equal to c_1, then the one term that cannot be produced is $(k+1)c_1$ and we must have $-(a+b) = (k+1)c_1$. Write $-a = \ell c_1$ and $-b = mc_1$ so that $\ell + m \equiv k+1$. We have that ℓ and m are not 1 or $p-1$. In the event that ℓ and m are larger than $k+1$ we have that $k+1 < \ell < p-1$ and hence there are $p-2-k \geqslant 1$ of the c_i such that $c_i = -c_1$. Certainly $-a = (p-\ell)(-c_1)$ and $-b = (p-m)(-c_1)$. As we are assuming $\ell > k+1$ we have $p - \ell < p-k-1$ and the same for m. Hence, in this case, if we rearrange so that the new c_1 is the minus of the original c_1, then we may assume we are in the opposite case where $\ell, m < k+1$.

Note that

$$
\begin{aligned}
&[L_a, L_b, \overbrace{L_{c_1}, \ldots, L_{c_1}}^{k}, \overbrace{L_{-c_1}, \ldots, L_{-c_1}}^{p-k-2}] \\
&\quad \subseteq [L_a, L_{c_1}, L_b, \overbrace{L_{c_1}, \ldots, L_{c_1}}^{k-1}, \overbrace{L_{-c_1}, \ldots, L_{-c_1}}^{p-k-2}] \\
&\qquad + [L_b, L_{c_1}, L_a, \overbrace{L_{c_1}, \ldots, L_{c_1}}^{k-1}, \overbrace{L_{-c_1}, \ldots, L_{-c_1}}^{p-k-2}].
\end{aligned}
$$

It is only necessary for us to prove that these latter two are trivial. The proofs are similar and so we only give the first.

By repeated applications of the Jacobi identity and making use of the fact that

$$[L_b, \overbrace{L_{c_1}, \dots, L_{c_1}}^{m}] = 0$$

we find that $[L_a, L_{c_1}, L_b, \overbrace{L_{c_1}, \dots, L_{c_1}}^{k-1}, \overbrace{L_{-c_1}, \dots, L_{-c_1}}^{p-k-2}]$ is contained in

$$\sum_{i=0}^{m-1} [L_a, \overbrace{L_{c_1}, \dots, L_{c_1}}^{k-i}, [L_b, \overbrace{L_{c_1}, \dots, L_{c_1}}^{i}], \overbrace{L_{-c_1}, \dots, L_{-c_1}}^{p-k-2}].$$

As $\ell = k + 1 - m = k - (m - 1)$ we have that

$$[L_a, \overbrace{L_{c_1}, \dots, L_{c_1}}^{k-(m-1)}] = 0$$

and hence all of the terms in the above sum are 0. This completes the proof of the lemma. □

This has a direct consequence for certain ideals of L. Recall that a homogeneous ideal M of L satisfies $M = M_1 + \cdots + M_{p-1}$ where $M_a \subseteq M \cap L_a$.

Lemma 4.2. *Let L be a $\mathbb{Z}_p$-graded Lie ring with $L_0 = 0$. If M and N are homogeneous ideals of L such that $M \subseteq C_L([N, N]) \cap L'$, then*

$$[M, \overbrace{N, \dots, N}^{p-2}] = 0.$$

Proof. It is enough to show that $[M_b, N_{a_1}, \dots, N_{a_{p-2}}] = 0$ for all choices of $b, a_1, \dots, a_{p-2}$ non-zero elements of $\mathbb{Z}_p$. Note that because $[M, [N, N]] = 0$ we know that this original term is contained in $[M_b, N_{\sigma(a_1)}, \dots, N_{\sigma(a_{p-2})}]$ for any permutation σ. Therefore, if $a_1, \dots, a_{p-2}$ produce $-b$, then the term must be 0. In the other case, however, each a_i is $\pm a_1$ by Lemma 2.4 and hence, by Lemma 4.1, $[L', N_{a_1}, \dots, N_{a_{p-2}}] = 0$. The result now follows as $M \subseteq L'$. □

Using the last lemma, we can prove that certain ideals are in a specified term of the upper central series of L. Note here that if M and N are homogeneous ideals, then $[M, N]$ is also a homogeneous ideal using a grading like that of Section 2. That is, take $[M, N]_a = \sum_{b+c=a} [M_b, N_c]$.

Lemma 4.3. *Let L be a $\mathbb{Z}_p$-graded Lie ring with $L_0 = 0$. If M is a homogeneous ideal of L and $L^{(s)} \subseteq C_L(M)$ and $M \subseteq L'$, then $M \subseteq Z_{(p-2)^s}(L)$.*

Proof. We will proceed by induction on s. Certainly $s = 0$ yields a true statement. With $s = 1$, Lemma 4.2 with $N = L$ yields that

$$[M, \overbrace{L, \ldots, L}^{p-2}] = 0$$

and hence $M \subseteq Z_{p-2}(L)$.

Assume that $s \geqslant 2$ and that we have proven the result for $s - 1$. Let $N = L^{(s-1)}$. N is a homogeneous ideal using the grading from Section 2. Also $M \subseteq C_L([N, N])$. Lemma 4.2 implies that

$$[M, \overbrace{N, \ldots, N}^{p-2}] = 0.$$

Set

$${}^{i}M = [M, \overbrace{N, \ldots, N}^{i}],$$

so that ${}^{0}M = M$ and each ${}^{i}M$ is a homogeneous ideal. Note that ${}^{p-3}M \subseteq Z(N)$ and $L^{(s-1)} \subseteq N$. So, by the induction hypothesis, we conclude ${}^{p-3}M \subseteq Z_{(p-2)^{s-1}}(L)$. Setting $\tilde{L} = L/{}^{p-3}M$ and repeating the argument, we obtain ${}^{p-4}\widetilde{M} \subseteq Z_{(p-2)^{s-1}}(\tilde{L})$ and hence ${}^{p-4}M \subseteq Z_{2(p-2)^{s-1}}(L)$. Repeating the argument gives $M = {}^{0}M \subseteq Z_{(p-2)(p-2)^{s-1}}(L)$ and hence $M \subseteq Z_{(p-2)^{s}}(L)$. This completes the proof. □

We are now prepared to give the proof of Theorem B. Notice that the bound given is equal to $\sum_{i=0}^{s-1}(p-2)^i$.

Proof of Theorem B. If $s = 1$, then the Lie algebra is abelian. Assume $s \geqslant 2$ and we know the result for $s - 1$. Set $M = L^{(s-1)}$. We know that M is a homogeneous ideal under the grading introduced in Section 2. By induction, L/M is nilpotent of class at most $\sum_{i=0}^{s-2}(p-2)^i$. Lemma 4.3 implies that $M \subseteq Z_{(p-2)^{s-1}}(L)$. Therefore the class of L is at most $\sum_{i=0}^{s-1}(p-2)^i$ and this completes the proof. □

One final note

In the case of the prime 11, the next open case, Theorem A involves $\lfloor \log_2(p-3) \rfloor$ and we would have a smaller bound for the derived length if we could replace $p - 3$ by $p - 4$. It happens that

$$[\overbrace{L_1, \ldots, L_1}^{p-3}]$$

is contained in $Z_3(L)$, see Lemma 1 of [3]. Therefore, $L_{p-3}^{(2^{p-5}-1)}$ is contained in $Z_3(L) + \langle L_{p-2}, L_{p-1}\rangle$ and hence $L^{(2^{p-5}+\lfloor \log_2(p-4)\rfloor)}$ is contained in $Z_3(L) + \langle L_{p-1}, L_{p-2}\rangle$. Therefore $L^{(2^{p-5}+\lfloor \log_2(p-4)\rfloor+2)}$ is contained in $Z_3(L)$. While this does not improve on Favaretto's computer-derived bound for the nilpotency class in the case of the prime 11, this does improve on the bound in Theorem C in the case of primes of the form $2^n + 3$.

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